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A Variational Principle for the Navier–Stokes Equation

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Motivated from Arnold's variational characterization of the Euler equation in terms of geodesic families of diffeomorphisms, a variational principle for the motion of incompressible viscous fluids is presented. A volume preserving diffusion process with drift velocity field subject to the Navier–Stokes equation is shown to extremize the energy functional of the fluid under a certain class of stochastic variations.

1. MOTIVATION

The problem of deriving various partial differential equations in hydrodynamics by means of variational principles for the action functional of the fluid has been of particular interest from both physical and mathematical points of view [1–3]. The Euler equation for the dry water (i.e., the perfect fluid) admits a natural variational derivation [1, 4, 5]. Indeed, Arnold characterized the Euler flow as a geodesic flow of diffeomorphisms which extremizes the energy functional of the fluid [1, 4, 5]. The situation changes drastically if the viscosity of the fluid is taken into account. Dissipative terms in the Navier–Stokes equation seems to refuse being derived through a variational scheme [2]. To overcome such difficulty of treating viscous (i.e., dissipative) forces of the fluid inherent in ordinary (functional) calculus of variations, some extended notions of calculus of variations are needed. Inoue and Funaki [6] were the first who considered such extension. They derived the Navier–Stokes equation by

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putting a white noise fluctuation into the Euler flow and taking the statistical average. Recently [7], we presented stochastic calculus of variations which extends ordinary calculus of variations to stochastic processes like semimartingales. Then, based on it, Nakagomi *et al.* [8] proposed a new functional variational scheme for the Navier–Stokes equation. There, however, the local pressure field of the fluid appeared in the action functional from the first, and this seems unsatisfactory since the local pressure should be a unknown function to be determined by the Navier–Stokes equation just as the velocity field is.

In this paper we will improve our previous work on this point and investigate the structure of the Navier–Stokes flow within the realm of stochastic calculus of variations. More precisely, the Navier–Stokes flow of incompressible viscous fluid will be characterized as a volume preserving diffusion process which makes the total energy of the fluid stationary under a certain class of stochastic variations.

Throughout the analysis, the principal role is played by an Itô (diffusion) process whose drift velocity field is subject to the Navier–Stokes equation. This elegant idea has been presented by Nelson [9] in his functional analysis of the finite energy Navier–Stokes flow. The present paper will be, following Inoue and Funaki, the second experiment to derive the equation of irreversible phenomena by combining the variational principle with the probabilistic idea.

2. Incompressible Viscous Fluid

Let $\mathscr{M}$ be a compact Riemannian manifold with dimensionality n which has no boundary. Suppose $\mathscr{M}$ is filled up with an incompressible viscous fluid. Each point $a \in \mathscr{M}$ represents the position of each fluid particle at the initial time $t = 0$. To look at the motion of the fluid globally, we introduce a class of diffusion processes on $\mathscr{M}$. A diffusion process on $\mathscr{M}$ is an $\mathscr{M}$-valued Markov process with continuous sample paths whose Kolmogorov infinitesimal generator is given by $\Delta + u$, where Δ is the Laplace–Beltrami operator and u is a time dependent vector field of class C^2 called the drift velocity field. For the notions and notations of differential geometry, see [10, 11]. Let $\mathscr{X}(\mathscr{M})$ be the totality of diffusion processes on $\mathscr{M}$ with common uniform initial distribution $|\mathscr{M}|^{-1}\, dV(a)$, where $|\mathscr{M}|$ is the total volume of $\mathscr{M}$ and $dV(a)$ is the invariant volume element of $\mathscr{M}$. Now we assume that the global motion of the viscous fluid is represented by a diffusion process $X = \{X_t | 0 \leqslant t < \infty\}$ belonging to $\mathscr{X}(\mathscr{M})$. The incompressibility condition claims that X has invariant measure $|\mathscr{M}|^{-1}\, dV$ or equivalently its drift velocity field u is divergence free $\delta u = \sum_{i=1}^{n} \nabla_i\, u^i = 0$, where δ is the coderivative and ∇ is the covariant derivative of $\mathscr{M}$. By the notion of volume preserving diffusion processes, we denote such diffusion processes. They form a subclass $\mathscr{X}_{\mathrm{vp}}(\mathscr{M})$ of $\mathscr{X}(\mathscr{M})$.

Now we introduce the total energy (or the energy functional) of the fluid by

$$\mathscr{E}(X) = \tfrac{1}{2}\int_0^T \int_{\mathscr{M}} \|u(x,t)\|^2 \, dV(x)\, dt$$
$$= \tfrac{1}{2}\sum_{i=1}^{n} \int_0^T \int_{\mathscr{M}} (u^i(x,t))^2 \, dV(x)\, dt \tag{1}$$

for any finite $T > 0$, where $(u^1, u^2, ..., u^n)$ are the components of u and $\|\cdot\|$ is the Riemannian norm of the tangent space of $\mathscr{M}$. Since u is continuous on $\mathscr{M}$ compact, the integral in the right-hand side of Eq. (1) is well defined. As X belongs to $\mathscr{X}_{\mathrm{vp}}(\mathscr{M})$, it has the invariant measure $|\mathscr{M}|^{-1}\, dV$. Therefore, Eq. (1) becomes

$$\mathscr{E}(X) = \tfrac{1}{2}\,|\mathscr{M}|\, E\left[\int_0^T \|u(X_t, t)\|^2 \, dt\right], \tag{2}$$

where $E[\cdot]$ denotes the mathematical expectation. It seems worthwhile to notice here that the total energy $\mathscr{E}(X)$ is proportional to the relative entropy of the diffusion process $X \in \mathscr{X}_{\mathrm{vp}}(\mathscr{M})$ with respect to a Wiener diffusion $W \in \mathscr{X}_{\mathrm{vp}}(\mathscr{M})$ with vanishing drift velocity, that is,

$$\mathscr{E}(X)/|\mathscr{M}| = -E\left[\log \frac{dP_T^X}{dP_T^W}(W)\right] = -\int_{\Omega_T(\mathscr{M})} \log \frac{dP_T^X}{dP_T^W}\, dP_T^W, \tag{3}$$

where P_T^X and P_T^W are probability measures on the space $\Omega_T(\mathscr{M})$ of continuous paths in $\mathscr{M}$ (i.e., $\Omega_T(\mathscr{M}) = C([0,T], \mathscr{M})$) induced respectively by X and W, and dP_T^W/dP_T^W is the Radon–Nikodym derivative. Equation (3) can be shown directly if we use the Girsanov formula to compute the Radon–Nikodym derivative in a local coordinate chart [12].

Before proceeding to the stochastic variational analysis with the variational principle, we emphasize the difference of the present variational formulation of the viscous fluid flow from that of the perfect fluid flow. As expected by physicists for nonconservative diffusion processes, the incorporation of randomness produces the viscous (i.e., dissipative) terms in the equations of motion. For conservative diffusion processes, this is no longer the case [13].

3. Stochastic Variation of Diffusion Processes

By the definition of the total energy of a diffusion process X in $\mathscr{X}_{\mathrm{vp}}(\mathscr{M})$, it is clear that the Wiener diffusion $W \in \mathscr{X}_{\mathrm{vp}}(\mathscr{M})$ has the minimum total energy, $\mathscr{E}(W) = 0$, which represents an incompressible viscous fluid without any macroscopic motion. This is a trivial case, and so we look for extremals which are not the global (or absolute) minimum. To do so we need to introduce the notion of stochastic variation of diffusion processes.

Let $X = \{X_t | 0 \leqslant t < \infty\}$ be a diffusion process belonging to $\mathscr{X}_{\mathrm{vp}}(\mathscr{M})$. In other words, X is an $\mathscr{M}$-valued Markov process with continuous sample paths whose Kolmogorov infinitesimal generator is given by $\Delta + u$, where the drift velocity field u is divergence free. We called X a Wiener diffusion in $\mathscr{X}_{\mathrm{vp}}(\mathscr{M})$ if $u = 0$ identically, and wrote it $W = \{W_t | 0 \leqslant t < \infty\}$. The sample paths of this Wiener diffusion W on $\mathscr{M}$ is known to be constructed by those of a Wiener process $B = \{B_t | 0 \leqslant t < \infty\}$ on a Euclidean space R^n through the following symmetric stochastic diffrential equations in a local coordinate chart.

$$dW_t^j = \sum_{a=1}^{n} E_t^{aj} \circ dB_t^a, \qquad j = 1, 2, ..., n, \tag{4}$$

$$dE_t^{aj} = \sum_{k,m=1}^{n} \Gamma_{km}^j(W_t) E_t^{ak} \circ dW_t^m, \qquad j = 1, 2, ..., n. \tag{5}$$

Here, the family $\{E^a\}_{a=1}^n$ forms a random moving coordinate frame along the Wiener diffusion W, Γ_{jk}^i's are the connection coefficients of the Riemannian manifold $\mathscr{M}$, and $\circ$ denotes the symmetric product of Fisk–Stratonovich. Once the sample paths of W in $\mathscr{X}_{\mathrm{vp}}(\mathscr{M})$ are known, it becomes an easy task to get those of a general diffusion process in $\mathscr{X}_{\mathrm{vp}}(\mathscr{M})$. They are given by the stochastic differential equations in a local coordinate chart

$$dX_t^j = u^j(X_t, t)\, dt + dW_t^j, \qquad j = 1, 2, ..., n. \tag{6}$$

Now we introduce the notion of stochastic variation of a diffusion process. Let $X = \{X_t | 0 \leqslant t < \infty\}$ be a diffusion process of particular interest in $\mathscr{X}_{\mathrm{vp}}(\mathscr{M})$ and v be a time-dependent divergence free (i.e., $\delta v = 0$) vector field of class C^2 on $\mathscr{M}$ such that $v(x, 0) = v(x, T) = 0$ for any $x \in \mathscr{M}$. For each t in the time interval $[0, T]$, we consider the mean square differential equations

$$\frac{dX_t^j(k)}{dk} = v^j(X_t(k), t), \qquad j = 1, 2, ..., n, \tag{7}$$

in a local coordinate chart. The real parameter k takes its value in the interval $(-K, K)$, $K > 0$, and the initial condition

$$X_t(0) = X_t \tag{8}$$

for each t is assumed. The t-indexed solutions $X_t(k)$, $-K < k < K$, form a family of random variables $\{X_t(k) \mid 0 \leqslant t \leqslant T, -K < k < K\}$ which will be called a stochastic variation (along the vector field v) of the diffusion process. It represents a continuous variation of the diffusion process X in the direction of v. By construction $\{X_t(0) \mid 0 \leqslant t \leqslant T\}$ coincides with the original diffusion process X in the interval $[0, T]$. For sufficiently small k, we have

$$X_t^j(k) = X_t^j + kv^j(X_t, t) + o(k), \qquad j = 1, 2, ..., n, \tag{9}$$

in a local coordinate chart. As X is a diffusion process and v is of class C^2, a stochastic process $X(k) = \{X_t(k) \mid 0 \leqslant t \leqslant T\}$ is a semimartingale adapted to X.

Let us return to the expression of the total energy of the fluid (1) or (2). Unfortunately, it is defined only for the volume preserving diffusion processes X's in $\mathscr{X}_{\mathrm{vp}}(\mathscr{M})$, and not for the stochastic processes $X(k)$'s, $k \neq 0$, consisting of the stochastic variation. This is because the stochastic variation $X(k)$ of the diffusion process X is not a diffusion process, but a more general semimartingale. However, the total energy of the fluid (2) has a natural extension which has a correct meaning as the total energy also for the stochastic variation $X(k)$'s. Let $\mathscr{P}_t$ be a sub-σ-algebra of events generated by $\{X_s \mid 0 \leqslant s \leqslant t\}$. Then $\mathscr{P} = \{\mathscr{P}_t \mid 0 \leqslant t < \infty\}$ is a natural forward filtration of the diffusion process X in $\mathscr{X}_{\mathrm{vp}}(\mathscr{M})$. The drift vector field u of X is given by the mean forward derivative of Nelson [13, 14]

$$DX_t = \lim_{h \downarrow 0} h^{-1} E[\tan(X_{t+h}, X_t) \mid \mathscr{P}_t] = u(X_t, t), \tag{10}$$

where $E[\cdot \mid \mathscr{P}_t]$ is the conditional expectation with respect to $\mathscr{P}_t$, and $\tan(X_{t+h}, X_t)$ for sufficiently small $h > 0$ denotes a vector in the tangent space of $\mathscr{M}$ at X_t tangent to the unique geodesic connecting X_t and X_{t+h} whose norm is equal to the length of the geodesic. Then the drift vector of the stochastic variation $X(k)$ would be defined by the mean forward derivative which makes sense even for the general semimartingales,

$$DX_t(k) = \lim_{h \downarrow 0} h^{-1} E[\tan(X_{t+h}(k), X_t(k)) \mid \mathscr{P}_t]. \tag{11}$$

The natural extension of the total energy of the fluid (2) is given by the functional

$$\tfrac{1}{2} |\mathscr{M}| \, E\left[\int_0^T \|DX_t\|^2 \, dt\right], \tag{12}$$

which makes sense also for the stochastic variation $X(k)$'s and will be denoted also by $\mathscr{E}(X)$.

To see the necessary and sufficient condition for the volume preserving diffusion process X being the extremal of the total energy $\mathscr{E}(X)$, we introduce a real function

$$m(k) = \tfrac{1}{2}E\left[\int_0^T \|DX_t(k)\|^2\, dt\right] = \mathscr{E}(X(k))/|\mathscr{M}| \tag{13}$$

defined on the interval $(-K, K)$. If the derivative $m'(k) = dm(k)/dk$ vanishes for $k = 0$, the total energy of the diffusion process $\mathscr{E}(X)$ is said to be stationary under the stochastic variation along v.

4. A Variational Principle

Having introduced the notion of stochastic variation of a diffusion process on the compact manifold $\mathscr{M}$, we present now a variational principle for the motion of an incompressible viscous fluid filling up the manifold $\mathscr{M}$.

Theorem 1.

$$m'(0) = -\int_0^T \int_{\mathscr{M}} ((A_* u)(x, t), v(x, t))\, dV(x)\, dt, \tag{14}$$

where A_* is a differential operator given by

$$A_* = \frac{\partial}{\partial t} + (u, d) - \Delta = \frac{\partial}{\partial t} + \sum_{i=1}^n u^i \nabla_i - \Delta, \tag{15}$$

d is the exterior derivative of $\mathscr{M}$ and $(\ ,\ )$ is the Riemannian inner product of the tangent space of $\mathscr{M}$.

Proof. A straightforward calculation gives

$$m'(0) = E\left[\int_0^T \left(DX_t(k), \frac{d}{dk} DX_t(k)\right) dt\right]\Bigg|_{k=0}. \tag{16}$$

Since D and d/dk commutes with eachother,

$$m'(0) = E\left[\int_0^T (DX_t, Dv(X_t, t))\, dt\right]. \tag{17}$$

As v is of class C^2, a stochastic process $v(X, t) = \{v(X_t, t) \mid 0 \leqslant t < \infty\}$ is indeed a semimartingale adapted to $\mathscr{P}$ and its mean forward derivative is

$$Dv(X_t, t) = (Av)(X_t, t), \tag{18}$$

where A is a differential operator given by

$$A = \frac{\partial}{\partial t} + (u, d) + \Delta = \frac{\partial}{\partial t} + \sum_{i=1}^{n} u^i \nabla_i + \Delta. \tag{19}$$

Notice that for any time-dependent vector field of class C^2, α, its mean forward derivative $D\alpha(X_t, t)$ along the diffusion process X is derived in terms of the stochastic parallel displacement discovered by Itô [15–17] as

$$D\alpha(X_t, t) = \lim_{h \downarrow 0} h^{-1} E[\Pi_{(t+h,t)} \alpha(X_{t+h}, t+h) - \alpha(X_t, t) \mid \mathscr{P}_t]. \tag{20}$$

Here, $\Pi_{(t+h,t)}$ is a random regular linear mapping from the tangent space of $\mathscr{M}$ at X_{t+h} onto that at X_t given by the symmetric stochastic differential equations in a local coordinate chart

$$d\beta_r^i = \sum_{j,q=1}^{n} \Gamma_{jq}^i(X_{t+r}) \beta_r^j \circ dX_{t+r}^q, \qquad 0 \leqslant r \leqslant h, \quad i = 1, 2, \ldots, n, \tag{21}$$

with initial condition $\beta_0 =$ a vector in the tangent space of $\mathscr{M}$ at X_t, say γ, in such a way as $\beta_h = \prod_{(t+h,t)} \gamma$ [17]. A simple stochastic calculus with (21) gives (18). Then Eq. (18) becomes

$$m'(0) = \int_0^T \int_{\mathscr{M}} (u(x, t), (Av)(x, t))\, dV(x)\, dt \tag{22}$$

which recovers the desired result (14), because $\delta u = 0$, $v(x, 0) = v(x, T) = 0$ for every $x \in \mathscr{M}$ and $\mathscr{M}$ is compact without boundary. Q.E.D.

THEOREM 2 (Variational principle). *A volume preserving diffusion process $X \in \mathscr{X}_{\mathrm{vp}}(\mathscr{M})$ has extremal total energy under the stochastic variations along any divergence free time-dependent C^2 vector field vanishing at $t = 0$ and T if and only if there exists a certain time-dependent real-valued function p and the drift velocity field u of X satisfies the Navier–Stokes equation*

$$\frac{\partial u}{\partial t} + (u, d)\, u - \Delta u - dp = 0, \qquad \delta u = 0. \tag{23}$$

Proof. Introduce a pre-Hilbert space of vector fields of class C^2 on $\mathscr{M}$, $H(T\mathscr{M}, dV)$, equipped with the inner product

$$\langle \alpha, \beta \rangle = \int_{\mathscr{M}} (\alpha(x), \beta(x))\, dV(x) \tag{24}$$

for $\alpha, \beta \in T\mathscr{M}$ (the tangent vector bundle of $\mathscr{M}$). Noticing $\delta v = 0$, use the variant of the Hodge decomposition theorem to get the gradient term $dp = \nabla p$ [5, 10]. By the standard technique (i.e., the Haar lemma [18]) of functional calculus of variations, Eq. (14) claims $m'(0) = 0$ if and only if

$$A_* u + dp = 0 \tag{25}$$

which is nothing but the Navier–Stokes equation. $\delta u = 0$ holds from the first since $X \in \mathscr{X}_{\mathrm{vp}}(\mathscr{M})$. Q.E.D.

Thus we have shown that the motion of the incompressible viscous fluid in $\mathscr{M}$ is represented by a volume preserving diffusion process which makes the total energy functional stationary under the stochastic variations. The drift vector field of this diffusion process is subject to the Navier–Stokes equation.

Remark 1. From the physical point of view, the diffusion process X whose drift velocity field is subject to the Navier–Stokes equation is a mesoscopic representation of the motion of the incompressible viscous fluid, whereas the usual differential dynamical system whose velocity field agrees with the above drift velocity field is its macroscopic representation in which the internal fluctuation of the fluid is neglected. Of course, a microscopic representation of the incompressible viscous fluid as a many body system of H_2O molecules will hardly provide us with a deep understanding of hydrodynamic phenomena. Interesting accounts on the idea of such conceptually different stages of methodology in natural phenomena are given by Zambrini [19]. A general theory of mesoscopic description of physical phenomena is given by Nakagomi [20].

Remark 2. Even if external forces are imposed on the fluid and are expressed as a gradient dV_{ex}, where V_{ex} is a given time-dependent real-valued function of class C^1 on $\mathscr{M}$, the total energy of the fluid appearing in the variational principle should not be modified, since V_{ex} can be absorbed in p.

Remark 3. If $\mathscr{M}$ has a smooth boundary $\partial\mathscr{M}$, then the direction v of the stochastic variation is assumed to vanish on $\partial\mathscr{M}$. Furthermore, the drift velocity field must vanish also on $\partial\mathscr{M}$. These boundary conditions are required to make the Laplace–Beltrami operator Δ symmetric on $H(T\mathscr{M}, dV)$.

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REFERENCES

1. V. ARNOLD, "Les Méthodes Mathématiques de la Mécanique Classique," Mir, Moscow, 1974.
2. W. YOURGRAU AND S. MANDELSTAM, "Variational Principles in Dynamics and Quantum Theory," Dover, New York, 1968.
3. J. T. ODEN AND J. N. REDDY, "Variational Methods in Theoretical Mechanics," Springer, Berlin, 1976.
4. V. ARNOLD, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, *Ann. Inst. Fourier* (*Grenoble*) **16** (1966), 319–361.
5. D. G. EBIN AND J. MARSDEN, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* **92** (1970), 102–163.
6. A. INOUE AND T. FUNAKI, On a new derivation of the Navier–Stokes equation, *Comm. Math. Phys.* **65** (1979), 83–90.
7. K. YASUE, Stochastic calculus of variations, *J. Funct. Anal.* **41** (1981), 327–340.
8. T. NAKAGOMI, K. YASUE, AND J.-C. ZAMBRINI, A stochastic variational derivation of the Navier–Stokes equation, *Lett. Math. Phys.* **5**, (1981), 545–552.
9. E. NELSON, Les écoulements incompressibles d'énergie finie, *in* "Colloques internationaux du CNRS," No. 117, pp. 159–165, Paris, 1962.
10. E. NELSON, "Tensor Analysis," Princeton Univ. Press, Princeton, N.J., 1967.
11. C. T. J. DODSON AND T. POSTON, "Tensor Geometry," Pitman, London, 1977.
12. W. H. FLEMING AND R. W. RISHEL, "Deterministic and Stochastic Optimal Control," Springer, Berlin, 1975.
13. E. NELSON, "Dynamical Theories of Brownian Motion," Princeton Univ. Press, Princeton N.J., 1967.
14. T. DANKEL, Mechanics on manifolds and the incorporation of spin into Nelson's stochastic mechanics, *Arch. Rational Mech. Anal.* **37** (1970), 192–221.
15. K. ITÔ, The Brownian motion and tensor fields on Riemannian manifold, *in* "Proc. Intern. Congress Mathematicians," pp. 536–539, Stockholm, 1962.
16. K. ITÔ, "Stochastic Parallel Displacement," Lecture Notes in Mathematics, No. 451, pp. 1–7, Springer, Berlin, 1975.
17. K. ITÔ, Extension of stochastic integrals, *in* "Proc. Intern. Symp. SDE, Kyoto 1976" (K. Itô, Ed.), Wiley, New York, 1978.
18. I. M. GELFAND AND S. V. FOMIN, "Calculus of Variations," Prentice–Hall, Englewood Cliffs, N.J., 1963.
19. J.-C. ZAMBRINI, "De l'incertitude: Positivisme et Sociologie," Anthropos, Paris, 1982.
20. T. NAKAGOMI, Mesoscopic thermodynamics of nonequilibrium open systems I, *J. Statist. Phys.* **26** (1981), 567–611.